Published by Institute of Physics Publishing for SISSA

Received: *July 18, 2005*
Revised: *September 1, 2005*
Accepted: *September 12, 2005*
Published: *October 3, 2005*

Complex matrix model and fermion phase space for bubbling AdS geometries

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Abstract: We study a relation between droplet configurations in the bubbling AdS geometries and a complex matrix model that describes the dynamics of a class of chiral primary operators in dual $\mathcal{N} = 4$ super Yang Mills (SYM). We show rigorously that a singlet holomorphic sector of the complex matrix model is equivalent to a holomorphic part of two-dimensional free fermions, and establish an exact correspondence between the singlet holomorphic sector of the complex matrix model and one-dimensional free fermions. Based on this correspondence, we find a relation of the singlet holomorphic operators of the complex matrix model to the Wigner phase space distribution. By using this relation and the AdS/CFT duality, we give a further evidence that the droplets in the bubbling AdS geometries are identified with those in the phase space of the one-dimensional fermions. We also show that the above correspondence actually maps the operators of $\mathcal{N} = 4$ SYM corresponding to the (dual) giant gravitons to the droplet configurations proposed in the literature.

Keywords: AdS-CFT and dS-CFT Correspondence, Matrix Models.

JHEP10(2005)004

Contents

1. Introduction

Recently much attention has been paid to the half-BPS sector in the AdS/CFT correspondence [1−3] due to the following two interesting observations. First, it was discussed in [4−6] that the dynamics of the chiral primary operators of $\mathcal{N} = 4$ super Yang Mills (SYM) on $S^3 \times R$ that correspond to Kaluza-Klein (KK) gravitons, giant gravitons [7, 8] and dual giant gravitons [9, 4] in the supergravity side is described by a complex matrix model (complex matrix quantum mechanics), which is obtained by reducing $\mathcal{N} = 4$ SYM to one dimension. It was pointed out in [6] that a holomorphic sector of the complex matrix model is related to one-dimensional ($(1+1)$-dimensional) fermions in the harmonic potential. In fact, the one-dimensional free fermions arise from a hermitean matrix model (hermitean matrix quantum mechanics) whose action is formally the same as that of the complex matrix model [11], and the number of the degrees of freedom of the hermitean matrix model is equal to that of the holomorphic sector of the complex matrix model. Second, as shown by Lin-Lunin-Maldacena (LLM) [10], the half-BPS solutions of type IIB supergravity that preserve $R \times \mathrm{SO}(4) \times \mathrm{SO}(4)$ isometry are characterized by a single function satisfying a differential equation. The single function can be determined by giving an appropriate boundary condition in a two-dimensional subspace in ten-dimensional spacetime. Giving a boundary condition is equivalent to specifying shapes of droplets on the two-dimensional subspace. This is so-called bubbling AdS geometries (for further developments, see [12]–[32]). The aforementioned gravity states should belong to the above class of the half-BPS solutions. It is plausible due to the first observation mentioned above that these droplets are identified with those in the phase space of the one-dimensional free fermions. In fact, some evidences that support this remarkable identification have been reported [10, 12, 13].

However, a precise map between the complex matrix model in the SYM side and the droplet configurations in the bubbling AdS geometries has not been given in the literature. The purpose of this paper is to find this map and give a further evidence of the above identification. Revisiting the analyses in [5, 14], we first show rigorously that the singlet holomorphic sector of the complex matrix model is equivalent to a holomorphic part of two-dimensional ($(2+1)$-dimensional) free fermions in the spherically symmetric harmonic potential. It has been discussed [33, 34] that these two-dimensional free fermions are related to the one-dimensional free fermions. Another interesting work relevant for us is ref. [13]. The authors of [13] identified fluctuations of the droplet of the $AdS_5 \times S^5$ geometry that are responsible for the KK gravitons. Then, by representing the effective action for the KK gravitons in terms of these fluctuations, they gave a supporting evidence of the identification. Motivated by these works, we next establish an exact correspondence between the singlet holomorphic sector of the complex matrix model and the one-dimensional free fermions. Based on this correspondence, we find a relation of the singlet holomorphic operators of the complex matrix model to the Wigner phase space distribution of the one-dimensional fermions. Using this relation and the AdS/CFT duality, we give a further evidence that the droplets in the bubbling AdS geometries are identified with those in the phase space of the one-dimensional free fermions. We also show that the above correspondence actually maps the operators of $\mathcal{N}=4$ SYM corresponding to the (dual) giant gravitons to the droplet configurations of the supergravity side proposed in [6, 10]. Most of our results also hold beyond the classical limit, namely at finite N, so that we expect to gain a clue from them in extending the bubbling AdS business to finite N.

The present paper is organized as follows. Section 2 is devoted to review. In section 2.1, we refer to a class of chiral primary operators of $\mathcal{N}=4$ SYM on $S^3 \times R$ (multi trace operators) and their correlation functions that are analogues of the extremal correlators of $\mathcal{N}=4$ SYM on R^4 and we are concerned with throughout the paper. We also describe a complex matrix model that is used in calculation of these correlation functions. In section 2.2, based on refs. [35, 12, 31], we summarize the results for the one-dimensional free fermions in the harmonic potential. In particular, we describe the classical phase space density and its quantum analogue, the Wigner phase space distribution. In section 2.3, we review the bubbling AdS geometries and its conjectured relation to the classical phase space of the one-dimensional free fermions. In section 3, we study the complex matrix model. We reduce the singlet holomorphic sector of the matrix model to the holomorphic part of two-dimensional free fermions in the spherically symmetric harmonic potential. We also develop the second quantization of these two-dimensional fermions. This gives an exact correspondence between the singlet holomorphic sector of the complex matrix model and the one-dimensional free fermions. In section 4, based on the correspondence, we find a relation of the singlet holomorphic operators of the complex matrix model to the Wigner phase space distribution. Using this relation and the AdS/CFT duality, we give a further evidence that the droplets in the bubbling AdS geometries are identified with those in the phase space of the one-dimensional fermions. In section 5, using the correspondence established in section 3, we construct in the Hilbert space of the one-dimensional fermions the states that correspond to the (dual) giant gravitons, and reproduce the droplet configurations of the (dual) giant gravitons in the supergravity side proposed in [6, 10]. Section 6 is devoted to summary and discussion. In appendix A, we present the results for harmonic oscillators in one and two dimensions, which we use in the main text. In appendix B, we give another derivation of the relation in section 4.

2. Review

2.1 Chiral primary operators and extremal correlators in $\mathcal{N} = 4$ SYM

In this subsection, we review some facts about a BPS sector of $\mathcal{N} = 4$ SYM. We are concerned with the chiral primary operators that take the form

$$\mathcal{O}^{\{J_1,J_2,\cdots,J_p\}} = \prod_{a=1}^{p} \mathrm{Tr}\left(Z^{J_a}\right), \tag{2.1}$$

where $Z = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$ with ϕ_1 and ϕ_2 being two of six real scalars of $\mathcal{N} = 4$ SYM. Note that these operators are singlet under U(N) transformations and holomorphic, namely depend only on Z. For simplicity, we denote a set $\{J_1, J_2, \cdots, J_p\}$ symbolically by $\{J\}$. We first consider correlation functions in $\mathcal{N} = 4$ SYM on R^4 of the type

$$\left\langle \left(\mathcal{O}^{\{J^{(0)}\}}(y)\right)^* \mathcal{O}^{\{J^{(1)}\}}(x_1)\cdots\mathcal{O}^{\{J^{(M)}\}}(x_M)\right\rangle, \qquad y, x_1, \ldots, x_M \in R^4\,, \tag{2.2}$$

which are called the extremal correlators in the literature. The charge conservation requires that

$$I^{(0)} = \sum_{r=1}^{M} I^{(r)}\,, \tag{2.3}$$

where

$$I^{(0)} = \sum_{a_0=1}^{p_0} J_{a_0}\,, \qquad I^{(r)} = \sum_{a_r=1}^{p_r} J_{a_r}\,. \tag{2.4}$$

There is a non-renormalization theorem on the extremal correlators telling that one can calculate them by using only the free part of the theory [36]. Next, we consider the chiral primary operators (2.1) in $\mathcal{N} = 4$ SYM on $S^3 \times R$. We restrict ourselves to the lowest Kaluza-Klein modes on S^3 because we are interested in half-BPS operators, so $\mathcal{O}^{\{J_1,J_2,\cdots,J_p\}}(t)$ has only the t-dependence. (Linear combinations of) these operators can represent the KK gravitons, the giant gravitons and the dual giant gravitons [5, 6, 10]. We

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consider correlation functions of the type

$$\left\langle \left(\mathcal{O}^{\{J^{(0)}\}}(t_0)\right)^* \mathcal{O}^{\{J^{(1)}\}}(t_1)\cdots\mathcal{O}^{\{J^{(M)}\}}(t_M)\right\rangle , \tag{2.5}$$

where $t_0 > t_r$ ($r = 1, \ldots, M$). These correlation functions are counterparts of the extremal correlators (2.2) on R^4. It is natural to consider that the non-renormalization theorem also holds for the correlation functions (2.5) and one can calculate them using the free part of the theory. Reducing $\mathcal{N} = 4$ SYM to the free part of $Z(t)$ yields a matrix quantum mechanics defined by

$$\begin{aligned}\mathcal{Z} &= \int [dZ(t)dZ^\dagger(t)]\, e^{iS} ,\\ S &= \int dt\, \mathrm{Tr}(\dot{Z}(t)\dot{Z}^\dagger(t) - Z(t)Z^\dagger(t)) ,\end{aligned} \tag{2.6}$$

where $Z(t)$ is an $N \times N$ complex matrix and the path-integral measure is defined through norm in the configuration space of the matrix

$$||dZ(t)||^2 = 2\,\mathrm{Tr}(dZ(t)dZ^\dagger(t)) . \tag{2.7}$$

The potential term in the action arises from a coupling of the conformal matter to the curvature of S^3 and we have rescaled the field and the time appropriately. In this paper, we assume the non-renormalization theorem for the correlation functions (2.5) and concentrate on their calculation through the complex matrix model (2.6). The condition $t_0 > t_r$ $(r = 1, \cdots, M)$ in (2.5) is naturally understood from the following fact, which was pointed out in [5]. The extremal correlators (2.2) calculated through the free part of the theory agree with the correlation functions (2.5) calculated through the matrix model if all the propagators $\langle Z_{ij}(x)Z^*_{kl}(y)\rangle = \delta_{ik}\delta_{jl}/(x-y)^2$ in (2.2) are replaced with those of the matrix model $\langle Z_{ij}(t)Z^*_{kl}(t_0)\rangle = \delta_{ik}\delta_{jl}e^{i(t-t_0)}/2$ that are valid for $t_0 > t$.

2.2 One-dimensional fermions in the harmonic potential

In this subsection, based on [35, 12, 31], we summarize the results for a one-dimensional system consisting of N non-interacting fermions in the harmonic potential. The classical one-body hamiltonian is given by $h_{cl} = \frac{1}{2}p^2 + \frac{1}{2}q^2$. First, we describe a classical aspect of this system. A useful object in discussing the classical aspect is the classical phase space density $u_{cl}(p, q, t)$, which takes the values 0 or 1. Regions in which $u_{cl}(p, q, t) = 1$ are called 'droplets'. The total area of droplets is equal to $2\pi\hbar N$ (we put $\hbar = 1$):

$$\int \frac{dpdq}{2\pi} u_{cl}(p, q, t) = N . \tag{2.8}$$

It is convenient to introduce the polar coordinates (r, ϕ),

$$q = r\cos\phi , \qquad p = r\sin\phi . \tag{2.9}$$

Then, $u_{cl}(p, q.t)$ is expressed as

$$u_{cl}(p, q, t) = \theta(\bar{r}(\phi) - r) , \tag{2.10}$$

where $\bar{r}(\phi)$ is the boundary profile function, namely the equation $r = \bar{r}(\phi)$ represents a shape of a boundary of a droplet. Here we assume that there is only a single droplet and $\bar{r}(\phi)$ is a single-valued function. For example, the ground state is represented by a circular droplet $\bar{r}(\phi) = r_0$ with $r_0^2 = 2N$. By using the classical equations of motion $\dot{q} = p$, $\dot{p} = -q$, it is easy to show that u_{cl} satisfies the equation of motion

$$\frac{\partial}{\partial t} u_{cl}(p,q,t) = \frac{\partial}{\partial \phi} u_{cl}(r\cos\phi, r\sin\phi, t) = \left(q\frac{\partial}{\partial p} - p\frac{\partial}{\partial q} \right) u_{cl}(p,q,t) \,. \tag{2.11}$$

Next, let us discuss a quantum aspect of the system. The quantum theory of the system is described by the second-quantized hamiltonian

$$\hat{\mathcal{H}} = \int dq \psi^\dagger(q,t) \left(-\frac{1}{2}\frac{\partial^2}{\partial q^2} + \frac{1}{2}q^2 \right) \psi(q,t) \tag{2.12}$$

and a constraint

$$\int dq \psi^\dagger(q,t)\psi(q,t) = N \,, \tag{2.13}$$

which turns out to fix the total number of particles to N. Here $\psi(q,t)$ is a fermionic field that satisfies the anti-commutation relation

$$\{\psi(q,t), \psi^\dagger(q',t)\} = \delta(q-q') \,. \tag{2.14}$$

$\psi(q,t)$ obeys the equation of motion

$$i\frac{\partial \psi(q,t)}{\partial t} = [\psi(q,t), \hat{\mathcal{H}}] = \left(-\frac{1}{2}\frac{\partial^2}{\partial q^2} + \frac{1}{2}q^2 \right) \psi(q,t) \,. \tag{2.15}$$

It is seen from (2.15) and (2.14) that $\psi(q,t)$ is expanded in terms of creation-annihilation operators $\hat{C}_n^\dagger$ and $\hat{C}_n$ with an anti-commutation relation $\{\hat{C}_m, \hat{C}_n^\dagger\} = \delta_{mn}$ as

$$\psi(q,t) = \sum_{n=0}^{\infty} \hat{C}_n e^{-iE_n t} \varphi_n(q) \,, \tag{2.16}$$

where $E_n = n + \frac{1}{2}$ and $\varphi_n(q)$ is the normalized one-body wave function of the n-th excited state, which is defined in (A.4). The hamiltonian (2.12) and the constraint (2.13) are expressed as

$$\begin{aligned} \mathcal{H} &= \sum_{n=0}^{\infty} E_n \hat{C}_n^\dagger \hat{C}_n \,, \\ \sum_{n=0}^{\infty} \hat{C}_n^\dagger \hat{C}_n &= N \,. \end{aligned} \tag{2.17}$$

The left hand side of the second equation is the number operator, as promised. The ground state is

$$|\Omega\rangle = \hat{C}_{N-1}^\dagger \hat{C}_{N-2}^\dagger \cdots \hat{C}_0^\dagger |0\rangle \,, \tag{2.18}$$

where $|0\rangle$ is the Fock vacuum defined by $\hat{C}_n|0\rangle = 0$ $(n = 0, 1, \cdots)$. Excited states are obtained by replacing some of the $\hat{C}_n^\dagger$ in $|\Omega\rangle$ with $\hat{C}_n^\dagger$'s with $n > N-1$ in such a way that the total number of $\hat{C}_n^\dagger$'s is preserved.

An object that plays a crucial role in our analysis is the Wigner phase space distribution $\hat{u}(p, q, t)$, which is defined by

$$\hat{u}(p, q, t) = \int dx e^{ipx} \psi^\dagger \left(q + \frac{x}{2}, t\right) \psi \left(q - \frac{x}{2}, t\right). \tag{2.19}$$

It follows from (2.15) and (2.13) that the Wigner phase space distribution satisfies the equation of motion

$$\frac{\partial}{\partial t}\hat{u}(p, q, t) = \left(q\frac{\partial}{\partial p} - p\frac{\partial}{\partial q}\right) \hat{u}(p, q, t) \tag{2.20}$$

and a constraint

$$\int \frac{dpdq}{2\pi}\hat{u}(p, q, t) = N\,. \tag{2.21}$$

There is another important constraint on the expectation value of $\hat{u}$, $\bar{u}(p, q, t) = \langle\Upsilon|$ $\hat{u}(p, q, t)|\Upsilon\rangle$, where $|\Upsilon\rangle$ is an arbitrary state that satisfies the constraint (2.13). $\bar{u}(p, q, t)$ satisfies a constraint

$$\bar{u} * \bar{u}(p, q, t) = \bar{u}(p, q, t)\,, \tag{2.22}$$

where $*$ stands for the Moyal product:

$$(A * B)(p, q) = A(p, q) e^{\frac{i}{2}\left(\frac{\overleftarrow{\partial}}{\partial q}\frac{\overrightarrow{\partial}}{\partial p} - \frac{\overleftarrow{\partial}}{\partial p}\frac{\overrightarrow{\partial}}{\partial q}\right)} B(p, q)\,. \tag{2.23}$$

It is sometimes convenient to consider a Fourier transform of $\hat{u}(p, q, t)$:

$$\tilde{u}(\alpha, \beta, t) = \int \frac{dpdq}{2\pi} e^{-i(p\beta - q\alpha)}\hat{u}(p, q, t). \tag{2.24}$$

It immediately follows from (2.20) and (2.21) that $\tilde{u}(\alpha, \beta, t)$ satisfies the equation of motion

$$\frac{\partial}{\partial t}\tilde{u}(\alpha, \beta, t) = \left(\beta\frac{\partial}{\partial \alpha} - \alpha\frac{\partial}{\partial \beta}\right) \tilde{u}(\alpha, \beta, t)\,, \tag{2.25}$$

and a constraint

$$\tilde{u}(0, 0, t) = N\,. \tag{2.26}$$

It is easy to show that $\tilde{u}(\alpha, \beta, t)$ also satisfies the W_∞ algebra

$$[\tilde{u}(\alpha, \beta, t), \tilde{u}(\alpha', \beta', t)] = 2i \sin\frac{1}{2}(\alpha\beta' - \alpha'\beta)\tilde{u}(\alpha + \alpha', \beta + \beta', t)\,. \tag{2.27}$$

In fact, $\tilde{u}(\alpha, \beta, t)$ is characterized by (2.25)–(2.27).

An important property of $\hat{u}(p, q, t)$ is that its expectation value $\bar{u}(p, q, t)$ is reduced to the classical phase space density $u_{cl}(p, q, t)$ in the classical limit. Actually, because the Moyal product becomes the ordinary product in the classical limit, the constraint (2.22) is reduced to an equation $(\bar{u})^2 = \bar{u}$, whose solution is $\bar{u} = 0$ or 1. The equation of motion (2.20) and the constraint (2.21) also hold for $\bar{u}$, and these equation and constraint are equivalent to (2.8) and (2.11).

2.3 Half-BPS geometries and free fermion droplets

In this subsection, we review the LLM description of the $AdS_5 \times S^5$ geometry and the half-BPS fluctuation modes in [10, 13].

All half-BPS geometries of type IIB supergravity which preserve the $R \times \mathrm{SO}(4) \times \mathrm{SO}(4)$ isometry are obtained by LLM [10]. These half-BPS geometries are given by

$$ds^2 = -h^{-2}\left[dt + \sum_{i=1}^{2} V_i dx^i\right]^2 + h^2\left[dy^2 + \sum_{i=1}^{2} dx^i dx^i\right] + ye^G d\Omega_3^2 + ye^{-G} d\widetilde{\Omega}_3^2\,, \quad (2.28)$$

$$\begin{aligned} h^{-2} &= 2y\cosh G\,, \qquad z = \frac{1}{2}\tanh G\,, \\ y\partial_y V_i &= \epsilon_{ij}\partial_j z\,, \qquad y(\partial_i V_j - \partial_j V_i) = \epsilon_{ij}\partial_y z\,, \\ B_t &= -\frac{1}{4}y^2 e^{2G}\,, \quad \widetilde{B}_t = -\frac{1}{4}y^2 e^{-2G}\,, \\ F &= dB_t \wedge (dt + V) + B_t dV + d\widehat{B}\,, \qquad \widetilde{F} = d\widetilde{B}_t \wedge (dt + V) + \widetilde{B}_t dV + d\widehat{\widetilde{B}}\,, \\ d\widehat{B} &= -\frac{1}{4}y^3 *_3 d\left(\frac{z + \frac{1}{2}}{y^2}\right), \qquad d\widehat{\widetilde{B}} = -\frac{1}{4}y^3 *_3 d\left(\frac{z - \frac{1}{2}}{y^2}\right), \end{aligned} \quad (2.29)$$

where a single function $z(x^1, x^2, y)$ obeys the differential equation

$$\partial_i\partial_i z + y\partial_y\left(\frac{\partial_y z}{y}\right) = 0\,, \quad (2.30)$$

and characterizes the solutions. The function $\frac{1}{2} - z$ on the $y = 0$ plane, which we denote by w, takes the values 0 or 1 and defines droplets as u_{cl} does. In the LLM description, the $AdS_5 \times S^5$ geometry is given by

$$\begin{aligned} z(r, y; r_0) &= \frac{r^2 - r_0^2 + y^2}{2\sqrt{(r^2 + r_0^2 + y^2)^2 - 4r^2 r_0^2}}\,, \\ V_r &= 0\,, \qquad V_\phi = -\frac{1}{2}\left[\frac{r^2 + r_0^2 + y^2}{\sqrt{(r^2 + r_0^2 + y^2)^2 - 4r^2 r_0^2}} - 1\right]. \end{aligned} \quad (2.31)$$

Here r_0 is identified with the AdS radius R through $r_0 = R^2 = \sqrt{2N}$ and (r, ϕ) are the polar coordinates of the x_1-x_2 plane. The droplet configuration $(w(x_1, x_2))$ is a circular droplet with the radius r_0 in the (x_1, x_2) plane (see figure 1).

In [13] a subclass of metric perturbation around the $AdS_5 \times S^5$ is discussed. Such perturbation is obtained by considering small ripples of the circular droplet. The boundary of the perturbed droplet in the polar coordinates is given by

$$\bar{r}(\phi) = r_0 + \delta r(\phi)\,. \quad (2.32)$$

$\delta r(\phi)$ is expanded in Fourier modes as

$$\delta r(\phi) = \sum_{n \neq 0} a_n e^{in\phi}\,, \qquad a_n^* = a_{-n} \quad (2.33)$$

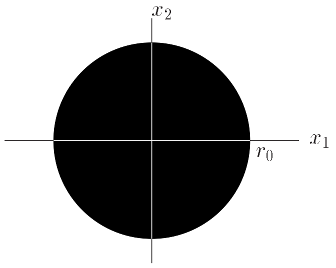

Figure 1: Circular droplet corresponding to $AdS_5 \times S^5$.

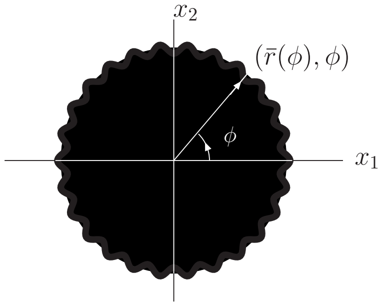

Figure 2: Fluctuation around the circular droplet.

(see figure 2). Here the zero mode is absent due to the area preserving deformation (to the first order in δr). Then one can express the resultant metric perturbation in terms of a_n. The result shows that the gravity modes described by a_n are the KK gravitons that, according to the standard AdS/CFT dictionary, correspond to $\mathrm{Tr}\, Z^n$ for $n > 0$ and $\mathrm{Tr}(Z^{\dagger|n|})$ for $n < 0$ in dual $\mathcal{N} = 4$ SYM. Moreover, one can read off the commutation relation among a_n's from the effective action that gives the equations of motion for this metric perturbation:

$$[a_m, a_n] \sim m\delta_{m+n} \,. \tag{2.34}$$

This commutation relation agrees with the one that is obtained when the droplet is regarded as the droplet on the classical phase space of one-dimensional fermions in the harmonic potential, with the identification $x_1 = q, \;\; x_2 = p$. Thus we are allowed to identify droplets in the $x_1 - x_2$ plane with those in the (p, q) phase space of the one-dimensional fermions.

3. Complex matrix model

3.1 Complex matrix model and free fermions in two dimensions

In this subsection, we study the complex matrix model introduced in section 2.2. In particular, we show that if we restrict ourselves to the correlation functions (2.5), we can regard the model as a holomorphic part of a two-dimensional system of free fermions in the spherically symmetric harmonic potential. The action in (2.6) is evaluated as

$$S = \int dt\, \mathrm{Tr}(\dot{Z}(t)\dot{Z}^\dagger(t) - Z(t)Z^\dagger(t)) = \int dt \sum_{i,j} (\dot{Z}(t)_{ij}\dot{Z}(t)^*_{ij} - Z(t)_{ij}(t)Z(t)^*_{ij}) \,, \tag{3.1}$$

while the norm (2.7) as

$$||dZ(t)||^2 = 2\,\mathrm{Tr}(dZ(t)dZ^\dagger(t)) = 2\sum_{i,j} dZ(t)_{ij}dZ(t)^*_{ij} \,, \tag{3.2}$$

from which we find an explicit form of the path-integral measure:

$$[dZ(t)dZ^\dagger(t)] = \left[2^{N^2} \prod_{i,j} d\,\mathrm{Re}\, Z(t)_{ij} d\,\mathrm{Im}\, Z(t)_{ij}\right] . \tag{3.3}$$

By comparing (3.1) and (3.3) with (A.15) and (A.16) in appendix A, we see that the system is nothing but a set of N^2 independent two-dimensional harmonic oscillators that are spherically symmetric.

We can quantize the system canonically as we do for the one-body system in appendix A. In what follows, we make use of the results in appendix A. The quantum hamiltonian is

$$\hat{H} = \sum_{i,j} \left(-\frac{\partial^2}{\partial Z_{ij} \partial Z_{ij}^*} + Z_{ij} Z_{ij}^* \right), \tag{3.4}$$

and the normalized ground state wave function is

$$\chi_0 = \frac{1}{\pi^{\frac{N^2}{2}}} e^{-\operatorname{Tr}(ZZ^\dagger)} = \frac{1}{\pi^{\frac{N^2}{2}}} e^{-\sum_{ij} Z_{ij} Z_{ij}^*}. \tag{3.5}$$

The measure for the inner product between the wave functions is

$$\int \prod_{ij} dZ_{ij} dZ_{ij}^* = 2^{N^2} \int \prod_{ij} d\operatorname{Re} Z_{ij} d\operatorname{Im} Z_{ij}. \tag{3.6}$$

We see from the results in appendix A that wave functions that take the form

$$\chi^{(J_1,\cdots,J_K)} = \left(\prod_{b=1}^{K} \operatorname{Tr}(Z^{J_b}) \right) \chi_0 \tag{3.7}$$

are eigenfunctions of $\hat{H}$ with eigenvalues $N^2 + \sum_{b=1}^{K} J_b$. This observation enables us to calculate the correlation functions (2.5) as follows:

$$\begin{aligned}
&\langle \left(\mathcal{O}^{\{J^{(0)}\}}(t_0) \right)^* \mathcal{O}^{\{J^{(1)}\}}(t_1) \cdots \mathcal{O}^{\{J^{(M)}\}}(t_M) \rangle = \\
&= \langle \chi_0 | \left(\mathcal{O}^{\{J^{(0)}\}}(t_0) \right)^* \mathcal{O}^{\{J^{(1)}\}}(t_1) \cdots \mathcal{O}^{\{J^{(M)}\}}(t_M) | \chi_0 \rangle \\
&= \langle \chi_0 | e^{i\hat{H}t_0} \left(\mathcal{O}^{\{J^{(0)}\}} \right)^* e^{-i\hat{H}(t_0-t_1)} \mathcal{O}^{\{J^{(1)}\}} e^{-i\hat{H}(t_1-t_2)} \cdots e^{-i\hat{H}(t_{M-1}-t_M)} \mathcal{O}^{\{J^{(M)}\}} e^{-i\hat{H}t_M} | \chi_0 \rangle \\
&= e^{i\sum_{r=1}^{M} I^{(r)}(t_r-t_0)} \int \prod_{ij} dZ_{ij} dZ_{ij}^* \left(\mathcal{O}^{\{J^{(0)}\}} \chi_0 \right)^* \left(\mathcal{O}^{\{J^{(1)}\}} \cdots \mathcal{O}^{\{J^{(M)}\}} \chi_0 \right).
\end{aligned} \tag{3.8}$$

Here the result is independent of the order of $t_1, \ldots, t_M$ although we assumed in the above calculation $t_0 > t_1 > \cdots > t_M$. Hence it is necessary and sufficient for calculating the correlation functions (2.5) to examine the wave functions (3.7), which are holomorphic except for the factor $e^{-\sum_{i,j} Z_{ij} Z_{ij}^*}$ and singlet under $\mathrm{U}(N)$ transformations.

Let us see that as far as the wave functions (3.7) are concerned, we can reduce the dynamical degrees of freedom of the system to N eigenvalues of Z. We express Z in terms of a unitary matrix and a triangle complex matrix as

$$Z = UTU^\dagger, \tag{3.9}$$

where $\mathrm{U}^\dagger = 1$, $T_{ij} = 0$ for $i > j$ and the T_{ii} $(i = 1, \ldots, N)$ are the eigenvalues of Z [37]. Note that the number of real parameters in Z is $2N^2$ while those in U and T are N^2 and $N(N+1)$, respectively. This reflects the fact that Z does not determine U and T uniquely:

if V is a unitary diagonal matrix, $(UV)^\dagger Z(UV) = V^\dagger TV$ and $V^\dagger TV$ is also a triangle matrix. This redundancy enables us to impose N constraints on variation of U. Variation of Z is

$$dZ = U(dT + i(dHT - TdH))U^\dagger \,, \tag{3.10}$$

where $dH = -iU^\dagger dU$ and dH is a hermitean matrix. We impose the constraint $dH_{ii} = 0$ $(i = 1, \ldots, N)$. Because $||dZ||^2 = ||dT + i(dHT - TdH)||^2$, we can make a change of variables from Z_{ij} to H_{ij} $(i > j)$ and T_{ij} $(i \le j)$. The jacobian for this change of variables is $|\Delta(z)|^2$ [37], where

$$\Delta(z) = \prod_{i<j}(z_i - z_j)\,, \qquad z_i \equiv T_{ii} \qquad (i = 1, \cdots, N)\,. \tag{3.11}$$

Hence we obtain

$$\int \prod_{i,j} dZ_{ij} dZ^*_{ij} = \int \prod_{i>j} dH_{ij} dH^*_{ij} \prod_{k<l} dT_{kl} dT^*_{kl} \prod_m dz_m dz^*_m |\Delta(z)|^2\,. \tag{3.12}$$

This leads us to redefine a wave function χ as

$$\chi_F \equiv \Delta(z)\chi \tag{3.13}$$

and the hamiltonian $\hat{H}$ as

$$\hat{H}_F \equiv \Delta(z)\hat{H}\frac{1}{\Delta(z)}\,. \tag{3.14}$$

Thus the system is equivalently described by the hamiltonian $\hat{H}_F$ and the wave functions χ_F with the inner product

$$\left\langle \chi_F^{(1)} | \chi_F^{(2)} \right\rangle = \int \prod_{i>j} dH_{ij} dH^*_{ij} \prod_{k<l} dT_{kl} dT^*_{kl} \prod_m dz_m dz^*_m \chi_F^{(1)*} \chi_F^{(2)}\,. \tag{3.15}$$

In particular, if χ is an eigenstate of $\hat{H}$ with an eigenvalue E, χ_F is an eigenstate of $\hat{H}_F$ with the same eigenvalue E. $\chi_F^{(J_1,\cdots,J_K)}$ is expressed in terms of z_i and T_{ij} $(i < j)$ as

$$\begin{aligned} \chi_F^{(J_1,\cdots,J_K)} &= \Delta(z)\chi^{(J_1,\cdots,J_K)} = \Delta(z)\left(\prod_{a=1}^{K}\sum_{i_a} z_{i_a}^{J_a}\right)\chi_0, \\ \chi_0 &= \frac{1}{\pi^{\frac{N^2}{2}}} e^{-\sum_i z_i z_i^* - \sum_{j<k} T_{jk} T^*_{jk}}\,. \end{aligned} \tag{3.16}$$

It should be noted that this eigenstate is independent of H_{ij} and H^*_{ij} due to the singlet nature of the wave function (3.7), while this eigenstate is independent of T_{ij}, T^*_{ij}, and z^*_i except for χ_0 due to the holomorphy and the singlet nature of the wave function (3.7). We see from this expression that $\chi_F^{(J_1,\cdots,J_K)}$ is equal to a certain linear combination of

$$\begin{aligned} &\begin{vmatrix} \Phi_{l_1}(z_1, z_1^*) & \Phi_{l_1}(z_2, z_2^*) & \cdots & \Phi_{l_1}(z_N, z_N^*) \\ \Phi_{l_2}(z_1, z_1^*) & & & \vdots \\ \vdots & & & \vdots \\ \Phi_{l_N}(z_1, z_1^*) & \cdots & \cdots & \Phi_{l_N}(z_N, z_N^*) \end{vmatrix} \times \prod_{j<k} \Phi_0(T_{jk}, T^*_{jk}) \\ &\text{with} \qquad \sum_i l_i = \frac{1}{2}N(N-1) + \sum_{b=1}^{K} J_b\,, \end{aligned} \tag{3.17}$$

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where $\Phi_l(z,z^*)$ is a 'holomorphic' wave function or a wave function of the lowest Landau level, which is defined in (A.13). Namely, $\chi_F^{(J_1,\cdots,J_K)}$ is a wave function of an energy eigenstate of a two-dimensional system consisting of N fermions and $\frac{1}{2}N(N-1)$ bosons which are governed by the spherically symmetric harmonic potential and do not interact each other. In this picture the coordinates of the fermions are represented by (z_i, z_i^*), while the coordinates of the bosons are represented by (T_{jk}, T_{ji}^*). Here, the fermions are in the "holomorphic" states, and the bosons are in the ground state. The total energy of the fermions is $\frac{1}{2}N(N+1)+\sum_{b=1}^{K} J_b$, where $\frac{1}{2}N(N+1)$ corresponds to the energy of the ground state of fermions, and the total energy of the boson is $\frac{1}{2}N(N-1)$. The sum of these two total energies is equal to the eigenvalue of $\hat{H}_F$ with respect to $\chi_F^{(J_1,\cdots,J_K)}$, $N^2+\sum_{b=1}^{K} J_b$, so that we can identify the system with the system of these free fermions and bosons, as far as we concentrate on the wave functions $\chi_F^{(J_1,\cdots,J_K)}$. Because the bosons are always in the ground state, their contribution to the eigenvalues of $\hat{H}_F$ is always $\frac{1}{2}N(N-1)$, and in the inner product $\langle \chi_F^{(J_1^{(1)},\cdots,J_{K_1}^{(1)})} | \chi_F^{(J_1^{(2)},\cdots,J_{K_2}^{(2)})} \rangle$ the integral over H_{ij} and T_{ij} gives an overall constant factor. Therefore we are allowed to consider only the fermionic part of $\chi_F^{(J_1,\cdots,J_K)}$ and replace the measure in the inner product (3.15) with $\int \prod_i dz_i dz_i^*$. In this way, we have reduced the singlet holomorphic sector of the complex matrix model to the holomorphic part of two-dimensional free fermions in the harmonic potential. This holomorphic nature enables us to relate these two-dimensional fermions to the one-dimensional ones, as we will see in the next subsection.

Finally we make a comment on the singlet non-holomorphic sector which is non-BPS. The hamiltonian (3.4) describes a set of N^2 non-interacting harmonic oscillators represented by (Z_{ij}, Z_{ij}^*). After change of variables from (Z_{ij}, Z_{ij}^*) to $(z_i, z_i^*, T_{ij}, T_{ij}^*, H_{ij}, H_{ij}^*)$, the hamiltonian is described by the dynamical degrees of freedom represented by $(z_i, z_i^*, T_{ij}, T_{ij}^*, H_{ij}, H_{ij}^*)$ that are complicatedly interacting. Nevertheless, as we showed above, the singlet holomorphic sector is reduced to the non-interacting fermions and bosons represented by (z_i, z_i^*) and (T_{ij}, T_{ij}^*) respectively. If we consider the singlet non-holomorphic sector, such simplification does not occur. We need to treat the above hamiltonian with the complicated interactions.

3.2 Second quantization of two-dimensional fermions

In this subsection, we develop a second quantized theory of the holomorphic part of the two-dimensional free fermions in the previous subsection and establish an exact correspondence between the singlet holomorphic sector of the complex matrix model and the one-dimensional free fermions. The observation in the previous subsection leads us to introduce a fermion field

$$\Psi(z,z^*,t)=\sum_{l=0}^{\infty}\hat{C}_l e^{-iE_l t}\Phi_l(z,z^*)\,, \qquad (3.18)$$

where $E_l = l+1$, and $\hat{C}_l$ and $\hat{C}_l^\dagger$ satisfy the anti-commutation relation $\{\hat{C}_l, \hat{C}_m^\dagger\} = \delta_{lm}$. The $\hat{C}_l$ in (3.18) will be identified with $\hat{C}_l$ in (2.16) below. This fermion field satisfies an anti-commutation relation

$$\{\Psi(z,z^*,t),\Psi^\dagger(z',z'^*,t)\} = \sum_{l=0}^{\infty}\Phi_l(z,z^*)\Phi_l(z',z'^*) \equiv K(z,z^*;z',z'^*)\,. \tag{3.19}$$

K behaves as the delta function with respect to functions spanned by the $\Phi_l(z,z^*)$, namely holomorphic functions of z times e^{-zz^*}. $\Psi(z,z^*,t)$ also satisfies a constraint that the total number of fermions be N,

$$\int dzdz^*\Psi^\dagger(z,z^*,t)\Psi(z,z^*,t) = \sum_{l=0}^{\infty}\hat{C}_l^\dagger\hat{C}_l = N\,. \tag{3.20}$$

The ground state $|\Omega\rangle$ is again given by

$$|\Omega\rangle = \hat{C}_{N-1}^\dagger\hat{C}_{N-2}^\dagger\cdots\hat{C}_1^\dagger\hat{C}_0^\dagger|0\rangle\,, \tag{3.21}$$

where $|0\rangle$ is the Fock vacuum defined by $\hat{C}_l|0\rangle = 0$.

We define the operators $\hat{s}_J(t)$ by

$$\hat{s}_J(t) = \int dzdz^* z^J\hat{U}(z,z^*,t)\,, \qquad \hat{U}(z,z^*,t) = \Psi^\dagger(z,z^*,t)\Psi(z,z^*,t)\,, \tag{3.22}$$

where $\hat{U}(z,z^*,t)$ is interpreted as the density operator restricted to the "holomorphic" sector. It is easy to verify that one can calculate the correlation functions (2.5) by making a correspondence in the second line of (3.8):

$$\begin{aligned}\mathrm{Tr}(Z(t)^J) &\leftrightarrow \hat{s}_J(t)\\ \mathrm{Tr}(Z^\dagger(t)^J) &\leftrightarrow \hat{s}_J^\dagger(t)\\ |\chi_0\rangle &\leftrightarrow |\Omega\rangle\,.\end{aligned} \tag{3.23}$$

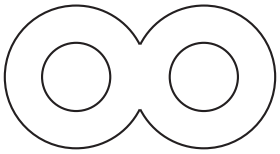
The explicit form of $\hat{s}_J(t)$ is easily calculated as

$$\hat{s}_J(t) = \frac{1}{2^{\frac{J}{2}}}e^{iJt}\sum_{l=0}^{\infty}\sqrt{\frac{(l+J)!}{l!}}\hat{C}_{l+J}^\dagger\hat{C}_l\,. \tag{3.24}$$

Note that because $[\hat{s}_J(t),\hat{s}_{J'}(t)] = 0$, there is no ordering ambiguity in translating the multi-trace operators into products of $\hat{s}_J$'s. Thus, if we identify $\hat{C}_l$ and $\hat{C}_l^\dagger$ in $\hat{s}_J$ with those of the one-dimensional fermions, we establish an exact correspondence between the singlet holomorphic sector of the complex matrix model and the one-dimensional free fermions in the harmonic potential.

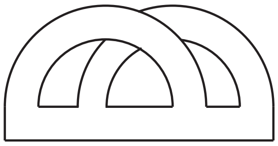
As an example, we give the calculation of $\langle\mathrm{Tr}(Z^\dagger(t_0)^4)\,\mathrm{Tr}(Z(t_1)^2)\,\mathrm{Tr}(Z(t_2)^2)\rangle$ with $t_0 > t_1,\ t_2$, based on the formalism we developed above.

$$\begin{aligned}\left\langle\mathrm{Tr}(Z^\dagger(t_0)^4)\,\mathrm{Tr}(Z(t_1)^2)\,\mathrm{Tr}(Z(t_2)^2)\right\rangle &= \langle\Omega|\hat{s}_4^\dagger(t_0)\hat{s}_2(t_1)\hat{s}_2(t_2)|\Omega\rangle\\ &= \frac{1}{2^4}e^{2i(t_1-t_0)+2i(t_2-t_0)}\langle\Omega|PQR|\Omega\rangle\,,\end{aligned} \tag{3.25}$$



$16N^3$

Figure 3: planar contribution.



$8N$

Figure 4: nonplanar contribution.

where

$$
\begin{aligned}
P = &\sqrt{(N+3)(N+2)(N+1)N}\hat{C}^\dagger_{N-1}\hat{C}_{N+3} + \sqrt{(N+2)(N+1)N(N-1)}\hat{C}^\dagger_{N-2}\hat{C}_{N+2} + \\
&+ \sqrt{(N+1)N(N-1)(N-2)}\hat{C}^\dagger_{N-3}\hat{C}_{N+1} + \sqrt{N(N-1)(N-2)(N-3)}\hat{C}^\dagger_{N-4}\hat{C}_{N} \\
Q = &\sqrt{(N-1)(N-2)}\hat{C}^\dagger_{N-1}\hat{C}_{N-3} + \sqrt{(N-2)(N-3)}\hat{C}^\dagger_{N-2}\hat{C}_{N-4} + \\
&+ \sqrt{(N+3)(N+2)}\hat{C}^\dagger_{N+3}\hat{C}_{N+1} + \sqrt{(N+2)(N+1)}\hat{C}^\dagger_{N+2}\hat{C}_{N}\,, \\
R = &\sqrt{(N+1)N}\hat{C}^\dagger_{N+1}\hat{C}_{N-1} + \sqrt{N(N-1)}\hat{C}^\dagger_{N}\hat{C}_{N-2}\,. \qquad (3.26)
\end{aligned}
$$

From (3.25), we obtain the final result

$$\frac{16N^3+8N}{2^4}e^{2i(t_1-t_0)+2i(t_2-t_0)}\,, \qquad (3.27)$$

which we can also obtain by applying the 'Wick theorem' with respect to Z_{ij} and Z^*_{ij} to the last line of (3.8). Note that the final result contains the non-planar contribution (See figures 3, 4).[1]

4. Singlet holomorphic operators and the Wigner phase space distribution

In this section, based on the exact correspondence between the singlet holomorphic sector of the complex matrix model and the one-dimensional fermions, which was established in the previous section, we find an exact relation of the singlet holomorphic operators of the complex matrix model to the Wigner phase space distribution of the one-dimensional fermions. The relation gives via the AdS/CFT duality an evidence that the droplets in the bubbling AdS geometries are identified with those in the phase space of the one-dimensional free fermions in the harmonic potential.

In section 2.3, we saw a_n $(n>0)$ in (2.33), which is the Fourier mode of the small fluctuation around the circular droplet (figure 2), yields the KK graviton that corresponds to $\mathrm{Tr}(Z(t)^n)$ in dual $\mathcal{N}=4$ SYM according to the standard AdS/CFT dictionary. In this case, w in section 2.3 that specifies the droplet is given by

$$
\begin{aligned}
w(x_1,x_2) &= \theta(r_0+\delta r(\phi)-r) \\
&= \theta(r_0-r)+\delta(r_0-r)\delta r(\phi)+\mathcal{O}((\delta r)^2)\,, \qquad (4.1)
\end{aligned}
$$

[1](3.27) is valid for U(N) gauge group. For SU(N), the subleading term is modified [38].

where $x_1 = r\cos\phi$, $x_2 = r\sin\phi$. Thus, we obtain from (2.33) a relation

$$a_n = \frac{1}{2\pi r_0 g_n(r_0)} \int dr \int d\phi r g_n(r) e^{-in\phi} w(x_1, x_2)\,. \tag{4.2}$$

At this stage, $g_n(r)$ is arbitrary. In the previous section, we constructed $\hat{s}_J(t)$ which corresponds to $\mathrm{Tr}(Z(t)^J)$ quantum mechanically. The quantum analogue of the classical phase space density u_{cl} is $\hat{u}$. Then, if the droplet specified by $w(x_1, x_2)$ are identified with that in the phase space of the free fermions through $w(x_1, x_2) = u_{cl}(p, q)$ with $x_1 = q$ and $x_2 = p$, we expect the following identity to hold quantum mechanically:

$$\hat{s}_J(t) = A_J \int dr \int d\phi r g_J(r) e^{-iJ\phi} \hat{u}(p, q, t)\,. \tag{4.3}$$

Here an overall coefficient A_J and a function $g_J(r)$ will be determined soon. The existence of the relation (4.3) serves as an evidence that the droplets in the bubbling AdS geometries are identified with those in the phase space of the one-dimensional free fermions. This is one of our results in this paper.

In what follows, we will see that the relation (4.3) indeed holds. We first evaluate $\hat{u}(p, q, t)$ explicitly. From (2.16) and (2.19), we obtain

$$\hat{u}(p, q, t) = \sum_{m,n=0}^{\infty} \hat{C}_m^\dagger \hat{C}_n e^{i(m-n)t} \int dx e^{ipx} \varphi_m^* \left(q + \frac{x}{2}\right) \varphi_n \left(q - \frac{x}{2}\right). \tag{4.4}$$

In order to calculate the integral in (4.4), we deform the contour as $x \to 2x + 2ip$ and make use of the following formula:

$$\int dx e^{-x^2} H_m(x + v) H_n(x + w) = 2^n \pi^{\frac{1}{2}} m! w^{n-m} L_m^{n-m}(-2vw) \qquad \text{for} \qquad m \le n\,, \tag{4.5}$$

where L_n^{n-m} is the Laguerre polynomial defined by

$$L_n^\alpha(x) = \sum_{m=0}^{n} (-1)^m \begin{pmatrix} n + \alpha \\ n - m \end{pmatrix} \frac{x^m}{m!}. \tag{4.6}$$

The final result is

$$\begin{aligned} \hat{u}(p, q, t) = & \sum_{m \le n} \hat{C}_m^\dagger \hat{C}_n e^{i(m-n)t} \sqrt{\frac{m!}{n!}} 2^{\frac{n-m}{2}+1} (-1)^m e^{i(m-n)\phi} e^{-r^2} r^{n-m} L_m^{n-m}(2r^2) + \\ & + \sum_{m > n} \hat{C}_m^\dagger \hat{C}_n e^{i(m-n)t} \sqrt{\frac{n!}{m!}} 2^{\frac{m-n}{2}+1} (-1)^n e^{i(m-n)\phi} e^{-r^2} r^{m-n} L_n^{m-n}(2r^2)\,. \end{aligned} \tag{4.7}$$

By comparing (3.24) with (4.3) and (4.7), we find that (4.3) holds if $A_J = \frac{1}{2\pi 2^J}$ and $g_J(r) = r^J$. Namely, we find

$$\begin{aligned} \hat{s}_J(t) &= \int dz dz^* z^J \hat{U}(z, z^*, t) \\ &= \frac{1}{2\pi 2^J} \int dr \int d\phi r^{J+1} e^{-iJ\phi} \hat{u}(p, q, t) \\ &= \int \frac{dp dq}{2\pi} \left(\frac{q - ip}{2}\right)^J \hat{u}(p, q, t). \end{aligned} \tag{4.8}$$

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From (4.8), we naively expect that a relation $\hat{U}(z, z^*, t) = \frac{1}{\pi}\hat{u}(p, q, t)$ holds with the identification $z = \frac{1}{2}(q - ip)$. Unfortunately, this is not the case. It is, however, known [34] that there is a simple relation between $\tilde{u}(\alpha, \beta, t)$ and a Fourier transform of $\hat{U}$. In appendix B, we give another derivation of (4.8) based on this relation.

5. Mapping (dual) giant gravitons to droplets

In this section, using the correspondence between the singlet holomorphic sector of the complex matrix model and the one-dimensional free fermions, which was established in section 3, we construct in the Hilbert space of the one-dimensional fermions the states corresponding to the (dual) giant gravitons. We will see that the classical phase space density that is the classical limit of the expectation value of the Wigner phase space distribution with respect to these states actually agree with the corresponding droplet configurations in the supergravity side proposed in [6, 10].

We first consider a sequence of non-negative integers $\langle\lambda\rangle = (\lambda_1, \lambda_2, \cdots, \lambda_N)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \geq 0$ and $\sum_{i=1}^N \lambda_i = n$. This sequence defines a Young tableau with the number of boxes in the i-th row being λ_i and the total number of boxes being n, and specifies an irreducible representation of $GL(N, C)$ as well as an irreducible representation of the symmetry group S_n. As in [5], we introduce an operator which is a linear combination of the chiral primary operators (2.1):

$$\chi_{\langle\lambda\rangle}(Z) = \frac{1}{n!}\sum_{\sigma\in S_n}\chi_{\langle\lambda\rangle}(\sigma)\,\mathrm{tr}(\sigma Z)\,, \tag{5.1}$$

where $\chi_{\langle\lambda\rangle}(Z)$ and $\chi_{\langle\lambda\rangle}(\sigma)$ are characters of the representations specified by $\langle\lambda\rangle$, and $\mathrm{tr}(\sigma Z)$ is defined by

$$\mathrm{tr}(\sigma Z) = \sum_{i_1, i_2, \cdots, i_n = 1}^{N} Z_{i_1 i_{\sigma(1)}} Z_{i_2 i_{\sigma(2)}} \cdots Z_{i_n i_{\sigma(n)}}\,. \tag{5.2}$$

As in [5], we make use of Weyl's character formula. Weyl's character formula tells that (5.1) is equal to the Schur polynomial:

$$\chi_{\langle\lambda\rangle}(Z) = \frac{\det_{i,j} z_j^{N-i+\lambda_i}}{\Delta(z)}\,. \tag{5.3}$$

Hence a fermionic wave function that is obtained by acting the operator (5.1) on the ground state is given by

$$\chi_F \sim \begin{vmatrix} \Phi_{N-1+\lambda_1}(z_1, z_1^*) & \Phi_{N-1+\lambda_1}(z_2, z_2^*) & \cdots & \Phi_{N-1+\lambda_1}(z_N, z_N^*) \\ \Phi_{N-2+\lambda_2}(z_1, z_1^*) & & & \vdots \\ \vdots & & & \vdots \\ \Phi_{\lambda_N}(z_1, z_1^*) & \cdots & \cdots & \Phi_{\lambda_N}(z_N, z_N^*) \end{vmatrix} \times \prod_{j<k}\Phi_0(T_{jk}, T_{jk}^*). \tag{5.4}$$

The correspondence established in section 3 maps this wave function to a state in the Hilbert space of the one-dimensional free fermions

$$\hat{C}^\dagger_{N-1+\lambda_1}\hat{C}^\dagger_{N-2+\lambda_2}\cdots\hat{C}^\dagger_{1+\lambda_{N-1}}\hat{C}^\dagger_{\lambda_N}|0\rangle\,. \tag{5.5}$$

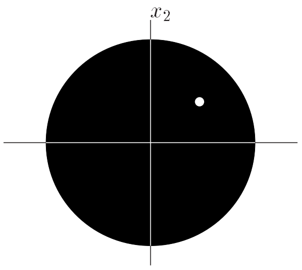

Figure 5: A giant graviton in $AdS_5 \times S^5$.

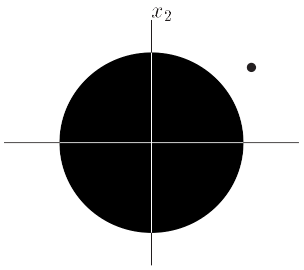

Figure 6: A dual giant graviton in $AdS_5 \times S^5$.

This state can also be obtained by rewriting (5.1) in terms of $\hat{s}_J$ through (3.23) and acting the resultant operator on the ground state $|\Omega\rangle$.

Next, let us construct in the Hilbert space of the one-dimensional fermions the states that represent the (dual) giant gravitons. In $\mathcal{N}=4$ SYM side, the giant graviton with the angular momentum L, where L is compatible with N, corresponds to the operator (5.1) with $\langle\lambda\rangle = (\lambda_1 = 1, \lambda_2 = 1, \cdots, \lambda_L = 1, \lambda_{L+1} = 0, \cdots, \lambda_N = 0)$ [8, 5]. This $\langle\lambda\rangle$ corresponds to the Young tableau that consists of a single column with L boxes. Note that there is a restriction $L \leq N$ and the giant graviton with $L = N$ is called the maximal giant graviton. We see from (5.5) that the state that is obtained by acting this operator on the ground state is mapped to a state in the Hilbert space of the one-dimensional fermions

$$|GG; L\rangle = \hat{C}^\dagger_N \hat{C}_{N-L} |\Omega\rangle \,. \tag{5.6}$$

The phase space density that is the classical limit of $\langle GG; L|\hat{u}(p,q,t)|GG; L\rangle$ obviously specifies the circular droplet with a small circular defect inside the circle (see figure 5). This droplet configuration is indeed proposed in [6, 10] as that of the giant graviton in the bubbling AdS geometries.

The dual giant graviton with angular momentum L, where L is compatible with or much larger than N, corresponds to the operator (5.1) with $\langle\lambda\rangle = (L, 0, \cdots, 0)$ [5]. This $\langle\lambda\rangle$ corresponds to the Young tableau that consists of a single row with L boxes. We again see from (5.5) that the state that is obtained by acting this operator on the ground state is mapped to a state in the Hilbert space of the one-dimensional fermions

$$|DGG; L\rangle = \hat{C}^\dagger_{N-1+L} \hat{C}_{N-1} |\Omega\rangle \,. \tag{5.7}$$

This is also consistent with the proposal in [6, 10] that the dual giant gravitons are given by the circular droplet and a small droplet outside the circular droplet in the bubbling AdS geometries (see figure 6).

6. Summary and discussion

In this paper, we study a relation between the droplet configurations in the bubbling AdS geometries and the singlet holomorphic sector of the complex matrix model that can describe a class of the chiral primary operators of $\mathcal{N}=4$ SYM. We first showed rigorously that the singlet holomorphic sector of the complex matrix model is equivalent to the holomorphic part of the two-dimensional fermions. We developed the second quantization of these fermions and established the exact correspondence between the singlet holomorphic sector of the complex matrix model and the one-dimensional free fermions in the harmonic potential. Next, based on this correspondence, we find a relation between the operators $\mathrm{Tr}(Z(t)^J)$ of the complex matrix model and the Wigner phase space distribution. This gives via the AdS/CFT duality a further evidence that the droplets in the bubbling AdS geometries are identified with those in the phase space of the one-dimensional free fermions. Finally, we also constructed the states corresponding to the (dual) giant gravitons in the Hilbert space of the one-dimensional fermions. We obtained the droplet configurations for these states, which are consistent with the proposals in [6, 10]. Our main results are summarized in (3.23), (3.24), (4.8), (5.6) and (5.7). These relations also hold at finite N, so that we expect to gain a clue from them in extending the bubbling AdS business to finite N.

We make a comment on a hermitean matrix model whose action is formally the same as the complex matrix model:

$$\mathcal{Z} = \int [d\phi(t)] e^{iS}\,, \qquad S = \int dt \left(\frac{1}{2}\dot{\phi}(t)^2 - \frac{1}{2}\phi(t)^2 \right), \tag{6.1}$$

where the path-integral measure is defined by the norm $||d\phi(t)||^2 = \mathrm{Tr}((d\phi(t))^2)$. As is well-known [11], as far as the $\mathrm{U}(N)$-invariant sector is concerned, the model can be reduced to a system of N eigenvalues and these eigenvalues behaves as one-dimensional non-interacting fermions in the harmonic potential, which were discussed in section 2.2. Then, a $\mathrm{U}(N)$-invariant operator $\mathrm{Tr}(\phi(t)^J)$ which is a counterpart of $\mathrm{Tr}(Z(t)^J)$ in the complex matrix model is translated to a operator in the second quantized theory of the fermions

$$\hat{v}_J(t) = \int dq q^J \psi^\dagger(q,t)\psi(q,t)\,. \tag{6.2}$$

We cannot, however, replace $\hat{s}_J$ with $\hat{v}_J$ in the calculation of the correlation functions (2.5). This yields extra terms that are formally obtained by 'contracting' Z_{ij} and Z_{kl}. This is a manifestation of the difference between the complex and hermitean matrix models.

There are some directions as extension of the present work. First, it has been discussed that the pp-wave limit of the bubbling AdS space is described by relativistic fermions in (1+1) dimensions [10, 30]. It is important to find a relation between these fermions and the complex matrix model. Topology changes in the bubbling geometry are also an interesting subject [29]. We hope our findings in this paper to give some insight to this subject.

Acknowledgments

We would like to thank Y. Susaki, T. Takayanagi and K. Yoshida for discussions. The work of Y.T. is supported in part by The 21st Century COE Program "Towards a New Basic Science; Depth and Synthesis." The work of A.T. is supported in part by Grant-in-Aid for Scientific Research (No. 16740144) from the Ministry of Education, Culture, Sports, Science and Technology.

A. Harmonic oscillators in one and two dimensions

In this appendix, we summarize the results for the harmonic oscillators in one and two dimensions, which we use throughout this paper. The harmonic oscillator in one dimension is defined by a hamiltonian,

$$\hat{h} = -\frac{1}{2}\frac{\partial^2}{\partial q^2} + \frac{1}{2}q^2 = \hat{a}^\dagger \hat{a} + \frac{1}{2}\,, \tag{A.1}$$

where $\hat{a}^\dagger$ and $\hat{a}$ are of course the creation and annihilation operators:

$$\hat{a} = \frac{1}{\sqrt{2}}\left(q + \frac{\partial}{\partial q}\right), \qquad \hat{a}^\dagger = \frac{1}{\sqrt{2}}\left(q - \frac{\partial}{\partial q}\right), \tag{A.2}$$

which satisfy the commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$. The ground state wave function $\varphi_0(q)$ is characterized by a relation $\hat{a}\phi_0(q) = 0$ and the normalized one takes the form

$$\varphi_0(q) = \frac{1}{\pi^{\frac{1}{4}}} e^{-\frac{1}{2}q^2}\,. \tag{A.3}$$

The normalized wave function of the n-th excited state, whose energy eigenvalue is $n + \frac{1}{2}$, is explicitly given by

$$\varphi_n(q) = \frac{1}{\sqrt{n!}}(\hat{a}^\dagger)^n \varphi_0(q) = \frac{1}{\sqrt{\pi^{\frac{1}{2}} 2^n n!}} H_n(q) e^{-\frac{1}{2}q^2}\,, \tag{A.4}$$

where H_n is the Hermite polynomial.

We generalize the above results to the two-dimensional spherically symmetric harmonic oscillator, which is defined by a hamiltonian,

$$\hat{h} = -\frac{1}{2}\frac{\partial^2}{\partial q_1^2} - \frac{1}{2}\frac{\partial^2}{\partial q_2^2} + \frac{1}{2}q_1^2 + \frac{1}{2}q_2^2 = \hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2 + 1\,, \tag{A.5}$$

where

$$\hat{a}_i = \frac{1}{\sqrt{2}}\left(q_i + \frac{\partial}{\partial q_i}\right), \qquad \hat{a}_i^\dagger = \frac{1}{\sqrt{2}}\left(q_i - \frac{\partial}{\partial q_i}\right), \qquad (i = 1, 2)\,. \tag{A.6}$$

These creation and annihilation operators satisfy the commutation relations

$$\left[\hat{a}_i, \hat{a}_j^\dagger\right] = \delta_{ij} \qquad [\hat{a}_i, \hat{a}_j] = \left[\hat{a}_i^\dagger, \hat{a}_j^\dagger\right] = 0\,. \tag{A.7}$$

The normalized wave function for the (m, n) excited state, whose energy eigenvalue is $m + n + 1$, is given by $\varphi_{m,n}(q_1, q_2) = \varphi_m(q_1)\varphi_n(q_2)$.

It is crucial for our analysis in the main text to introduce the complex coordinate:

$$z = \frac{1}{\sqrt{2}}(q_1 + i q_2)\,. \tag{A.8}$$

In this variable, the hamiltonian takes the form

$$\hat{h} = -\frac{\partial^2}{\partial z \partial z^*} + z z^* = \hat{c}_1^\dagger \hat{c}_1 + \hat{c}_2^\dagger \hat{c}_2 + 1\,, \tag{A.9}$$

where

$$\hat{c}_1 = \frac{1}{\sqrt{2}}(\hat{a}_1 + i\hat{a}_2) = \frac{1}{\sqrt{2}}\left(z + \frac{\partial}{\partial z^*}\right), \qquad \hat{c}_1^\dagger = \frac{1}{\sqrt{2}}\left(\hat{a}_1^\dagger - i\hat{a}_2^\dagger\right) = \frac{1}{\sqrt{2}}\left(z^* - \frac{\partial}{\partial z}\right),$$
$$\hat{c}_2 = \frac{1}{\sqrt{2}}(\hat{a}_1 - i\hat{a}_2) = \frac{1}{\sqrt{2}}\left(z^* + \frac{\partial}{\partial z}\right), \qquad \hat{c}_2^\dagger = \frac{1}{\sqrt{2}}\left(\hat{a}_1^\dagger + i\hat{a}_2^\dagger\right) = \frac{1}{\sqrt{2}}\left(z - \frac{\partial}{\partial z^*}\right), \quad (A.10)$$

which satisfy again the commutation relations $[\hat{c}_i, \hat{c}_j^\dagger] = \delta_{ij}$, $[\hat{c}_i, \hat{c}_j] = [\hat{c}_i^\dagger, \hat{c}_j^\dagger] = 0$. By acting the creation operators $\hat{c}_1^\dagger$ and $\hat{c}_2^\dagger$ successively on the ground state, one can construct energy eigenstates, which form a normalized orthonormal basis:

$$\Phi_{k,l}(z, z^*) = \frac{1}{\sqrt{k!l!}}\left(\hat{c}_1^\dagger\right)^k \left(\hat{c}_2^\dagger\right)^l \varphi_{0,0}\,. \qquad (A.11)$$

The ground state, whose energy eigenvalue is one, takes the form in the complex variable

$$\Phi_{0,0}(z, z^*) = \frac{1}{\sqrt{\pi}} e^{-zz^*}\,. \qquad (A.12)$$

The excited state $\Phi_{k,l}(z, z^*)$ with the energy eigenvalue $k + l + 1$ is expressed as a linear combination of $\varphi_{m,n}$ with $m + n = k + l$. In particular, $\Phi_{0,l}$ is holomorphic except for the factor e^{-zz^*}:

$$\Phi_{0,l}(z, z^*) = \sqrt{\frac{2^l}{\pi l!}} z^l e^{-zz^*}\,. \qquad (A.13)$$

We denote $\Phi_{0,l}(z, z^*)$ simply by $\Phi_l(z, z^*)$. $\Phi_l(z, z^*)$ is a wave function of the lowest Landau level in the context of the quantum Hall effect. The measure used for the inner product between the wave functions is

$$\int dq_1 dq_2 = 2\int d\,\mathrm{Re}\, z d\,\mathrm{Im}\, z^* \equiv \int dz dz^*\,. \qquad (A.14)$$

The system can be quantized equivalently by path-integral. The partition function is given by

$$Z = \int [dz(t)dz^*(t)]\, e^{-S}\,, \qquad S = \int dt(\dot{z}(t)\dot{z}^*(t) - z(t)z^*(t))\,, \qquad (A.15)$$

where

$$[dz(t)dz^*(t)] = [2d\,\mathrm{Re}\, z(t) d\,\mathrm{Im}\, z(t)]\,. \qquad (A.16)$$

B. Another derivation of (4.8)

In this appendix, we give another derivation of (4.8). We first consider a Fourier transform of $\hat{U}(z, z^*, t)$

$$\tilde{U}(\Lambda, \Lambda^*, t) = \int dz dz^* e^{\Lambda^* z - \Lambda z^*} \hat{U}(z, z^*, t)\,, \qquad (B.1)$$

which is an analogue of $\tilde{u}$. By substituting (3.18), we obtain

$$\tilde{U}(\Lambda, \Lambda^*, t) = \sum_{m,n=0}^{\infty} \hat{C}_m^\dagger \hat{C}_n e^{i(m-n)t} \sqrt{\frac{2^{m+n}}{m!n!}} (-1)^m \frac{\partial^{m+n}}{\partial \Lambda^m \partial \Lambda^{*n}} e^{-\frac{1}{2}\Lambda\Lambda^*}\,. \qquad (B.2)$$

Some algebra gives

$$
\begin{aligned}
&[\tilde{U}(\Lambda,\Lambda^*,t),\tilde{U}(\Lambda',\Lambda'^*,t)] = \\
&\quad = \left(e^{\frac{1}{2}\Lambda^*\Lambda'} - e^{\frac{1}{2}\Lambda\Lambda'^*}\right)\tilde{U}\left(\Lambda+\Lambda',\Lambda^*+\Lambda'^*,t\right) \\
&\quad = e^{\frac{1}{4}(\Lambda+\Lambda')(\Lambda^*+\Lambda'^*)-\frac{1}{4}\Lambda\Lambda^*-\frac{1}{4}\Lambda'\Lambda'^*} 2i\sin\frac{1}{4}(\Lambda^*\Lambda'-\Lambda\Lambda'^*)\tilde{U}(\Lambda+\Lambda',\Lambda^*+\Lambda'^*,t)\,. \qquad \text{(B.3)}
\end{aligned}
$$

This result urges us to define a new object

$$\tilde{U}_{\rm new}(\Lambda,\Lambda^*,t) = e^{\frac{1}{4}\Lambda\Lambda^*}\tilde{U}(\Lambda,\Lambda^*,t)\,. \qquad \text{(B.4)}$$

In fact, $\tilde{U}_{new}(\Lambda,\Lambda^*,t)$ satisfies the W_∞ algebra (2.27) with the identification $\Lambda=\beta-i\alpha$. It is also easy to see that under this identification $\tilde{U}_{new}(\Lambda,\Lambda^*,t)$ satisfies the same equation of motion (2.25) and constraint (2.26) as $\tilde{u}(\alpha,\beta,t)$ satisfies. Therefore $\tilde{U}_{new}$ would coincide with $\tilde{u}$:

$$\tilde{u}(\alpha,\beta,t) = e^{\frac{1}{4}\Lambda\Lambda^*}\tilde{U}(\Lambda,\Lambda^*,t)\,. \qquad \text{(B.5)}$$

We also checked this by explicit calculation. The algebra (B.3) and the relation (B.5) were already derived in [34] in the context of the integer quantum Hall effect. Thus we obtain a direct relation between $\hat{U}$ and $\hat{u}$:

$$\hat{U}(z,z^*,t) = \int\frac{d\Lambda d\Lambda^*}{4\pi^2}e^{-\Lambda^*z+\Lambda z^*-\frac{1}{4}\Lambda\Lambda^*}\int\frac{dpdq}{2\pi}e^{-\frac{\Lambda}{2}(q+ip)+\frac{\Lambda^*}{2}(q-ip)}\hat{u}(p,q,t)\,. \qquad \text{(B.6)}$$

Using (B.6), we calculate $\hat{s}_J(t)$ as

$$
\begin{aligned}
\hat{s}_J(t) &= \int dzdz^* z^J\hat{U}(z,z^*,t) \\
&= \int dzdz^*\int\frac{d\Lambda d\Lambda^*}{4\pi^2}(-1)^J\frac{\partial^J}{\partial\Lambda^{*J}}(e^{-\Lambda^*z+\Lambda z^*})e^{-\frac{1}{4}\Lambda\Lambda^*}\times \\
&\quad\times\int\frac{dpdq}{2\pi}e^{-\frac{\Lambda}{2}(q+ip)+\frac{\Lambda^*}{2}(q-ip)}\hat{u}(p,q,t) \\
&= \frac{\partial^J}{\partial\Lambda^{*J}}\left(e^{-\frac{1}{4}\Lambda\Lambda^*}\int\frac{dpdq}{2\pi}e^{-\frac{\Lambda}{2}(q+ip)+\frac{\Lambda^*}{2}(q-ip)}\hat{u}(p,q,t)\right)\Bigg|_{\Lambda=\Lambda^*=0}\,. \qquad \text{(B.7)}
\end{aligned}
$$

The last expression actually gives (4.8).

References

[1] J.M. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [hep-th/9711200].

[2] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from non-critical string theory*, *Phys. Lett.* **B 428** (1998) 105 [hep-th/9802109].

[3] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [hep-th/9802150].

[4] A. Hashimoto, S. Hirano and N. Itzhaki, *Large branes in AdS and their field theory dual*, *JHEP* **08** (2000) 051 [hep-th/0008016].

[5] S. Corley, A. Jevicki and S. Ramgoolam, *Exact correlators of giant gravitons from dual $N = 4$ SYM theory*, *Adv. Theor. Math. Phys.* **5** (2002) 809 [hep-th/0111222].

[6] D. Berenstein, *A toy model for the AdS/CFT correspondence*, *JHEP* **07** (2004) 018 [hep-th/0403110].

[7] J. McGreevy, L. Susskind and N. Toumbas, *Invasion of the giant gravitons from anti-de Sitter space*, *JHEP* **06** (2000) 008 [hep-th/0003075].

[8] V. Balasubramanian, M. Berkooz, A. Naqvi and M.J. Strassler, *Giant gravitons in conformal field theory*, *JHEP* **04** (2002) 034 [hep-th/0107119].

[9] M.T. Grisaru, R.C. Myers and O. Tafjord, *SUSY and Goliath*, *JHEP* **08** (2000) 040 [hep-th/0008015].

[10] H. Lin, O. Lunin and J. Maldacena, *Bubbling AdS space and 1/2 BPS geometries*, *JHEP* **10** (2004) 025 [hep-th/0409174].

[11] E. Brezin, C. Itzykson, G. Parisi and J.B. Zuber, *Planar diagrams*, *Commun. Math. Phys.* **59** (1978) 35.

[12] G. Mandal, *Fermions from half-BPS supergravity*, *JHEP* **08** (2005) 052 [hep-th/0502104].

[13] L. Grant, L. Maoz, J. Marsano, K. Papadodimas and V.S. Rychkov, *Minisuperspace quantization of 'bubbling AdS' and free fermion droplets*, *JHEP* **08** (2005) 025 [hep-th/0505079].

[14] A. Ghodsi, A.E. Mosaffa, O. Saremi and M.M. Sheikh-Jabbari, *LLL vs. LLM: half BPS sector of $N = 4$ SYM equals to quantum hall system*, hep-th/0505129.

[15] M.M. Caldarelli and P.J. Silva, *Giant gravitons in AdS/CFT, I. Matrix model and back reaction*, *JHEP* **08** (2004) 029 [hep-th/0406096].

[16] J.T. Liu, D. Vaman and W.Y. Wen, *Bubbling 1/4 BPS solutions in type-IIB and supergravity reductions on $S^n \times S^n$*, hep-th/0412043.

[17] D. Martelli and J.F. Morales, *Bubbling AdS_3*, *JHEP* **02** (2005) 048 [hep-th/0412136].

[18] N.V. Suryanarayana, *Half-BPS giants, free fermions and microstates of superstars*, hep-th/0411145.

[19] Z.W. Chong, H. Lu and C.N. Pope, *BPS geometries and AdS bubbles*, *Phys. Lett.* **B 614** (2005) 96 [hep-th/0412221].

[20] J.T. Liu and D. Vaman, *Bubbling 1/2 BPS solutions of minimal six-dimensional supergravity*, hep-th/0412242.

[21] O. Lunin and J. Maldacena, *Deforming field theories with* $\mathrm{U}(1) \times \mathrm{U}(1)$ *global symmetry and their gravity duals*, *JHEP* **05** (2005) 033 [hep-th/0502086].

[22] S. Mukhi and M. Smedback, *Bubbling orientifolds*, *JHEP* **08** (2005) 005 [hep-th/0506059].

[23] V. Filev and C.V. Johnson, *Operators with large quantum numbers, spinning strings and giant gravitons*, *Phys. Rev.* **D 71** (2005) 106007 [hep-th/0411023].

[24] H. Ebrahim and A.E. Mosaffa, *Semiclassical string solutions on 1/2 BPS geometries*, *JHEP* **01** (2005) 050 [hep-th/0501072].

[25] M.M. Sheikh-Jabbari, *Tiny graviton matrix theory: DLCQ of IIB plane-wave string theory, a conjecture*, *JHEP* **09** (2004) 017 [hep-th/0406214].

JHEP10(2005)004

[26] M.M. Sheikh-Jabbari and M. Torabian, *Classification of all 1/2 BPS solutions of the tiny graviton matrix theory*, *JHEP* **04** (2005) 001 [hep-th/0501001].

[27] A. Buchel, *Coarse-graining 1/2 BPS geometries of type-IIB supergravity*, hep-th/0409271.

[28] M.M. Caldarelli, D. Klemm and P.J. Silva, *Chronology protection in anti-de Sitter*, *Class. and Quant. Grav.* **22** (2005) 3461 [hep-th/0411203].

[29] P. Hořava and P.G. Shepard, *Topology changing transitions in bubbling geometries*, *JHEP* **02** (2005) 063 [hep-th/0502127].

[30] Y. Takayama and K. Yoshida, *Bubbling 1/2 BPS geometries and Penrose limits*, hep-th/0503057.

[31] A. Dhar, *Bosonization of non-relativstic fermions in 2-dimensions and collective field theory*, *JHEP* **07** (2005) 064 [hep-th/0505084].

[32] G. Milanesi and M. O'Loughlin, *Singularities and closed time-like curves in type-IIB 1/2 BPS geometries*, hep-th/0507056.

[33] S. Iso, D. Karabali and B. Sakita, *One-dimensional fermions as two-dimensional droplets via Chern-Simons theory*, *Nucl. Phys.* **B 388** (1992) 700 [hep-th/9202012].

[34] S. Iso, D. Karabali and B. Sakita, *Fermions in the lowest Landau level: bosonization, w(infinity) algebra, droplets, chiral bosons*, *Phys. Lett.* **B 296** (1992) 143 [hep-th/9209003].

[35] A. Dhar, G. Mandal and S.R. Wadia, *Classical fermi fluid and geometric action for* $c = 1$, *Int. J. Mod. Phys.* **A 8** (1993) 325 [hep-th/9204028]; *Nonrelativistic fermions, coadjoint orbits of* W_∞ *and string field theory at* $c = 1$, *Mod. Phys. Lett.* **A 7** (1992) 3129 [hep-th/9207011]; W_∞ *coherent states and path integral derivation of bosonization of nonrelativistic fermions in one-dimension*, *Mod. Phys. Lett.* **A 8** (1993) 3557 [hep-th/9309028]; *A time dependent classical solution of* $c = 1$ *string field theory and nonperturbative effects*, *Int. J. Mod. Phys.* **A 8** (1993) 3811 [hep-th/9212027].

[36] B.U. Eden, P.S. Howe, A. Pickering, E. Sokatchev and P.C. West, *Four-point functions in* $N = 2$ *superconformal field theories*, *Nucl. Phys.* **B 581** (2000) 523 [hep-th/0001138]; *Extremal and next-to-extremal n-point correlators in four- dimensional sCFT*, *Phys. Lett.* **B 494** (2000) 141 [hep-th/0004102].

[37] See, for example, M.L. Mehta, *Random matrices*, 2nd edition, Academic Press, New York, 1991.

[38] R. de Mello Koch and R. Gwyn, *Giant graviton correlators from dual* $\mathrm{SU}(N)$ *super Yang-Mills theory*, *JHEP* **11** (2004) 081 [hep-th/0410236].